Abstract— In this paper, Model Predictive Control (MPC) extended to a subclass of non-linear systems based on the subspace approach using the bilinear model. The merit of this approach is that MPC controller design is obtained directly from subspace matrices, which are identified based on system input/output data. This method is useful for unknown systems. Linear controllers cannot describe the practical systems because the practical systems are non-linear, so we proposed the MPC controller design based on the bilinear subspace model.

Keywords— NMPC; Bilinear Model; Subspace Identification

I. INTRODUCTION

The development of control systems involves two essential steps. The first step is system modeling, and the second step is a controller design based on the system model. One of the approaches to obtaining system models is available in the literature [1]. In this method, we can construct mathematical models of the system from observed input-output data that is called system identification.

Subspace identification is known as a nonparametric model identification method. The nonparametric model identification compared to the parametric model identification, because of higher variance error and less model structure has fewer bias errors. Also, it should be noted that by increasing the number of data, the problem of higher variance can be solved [2].

The feature of these methods, that could be defined as advantages, are based on reliable numerical algorithms of the QR factorization and the singular value decomposition (SVD), in theses algorithm, we do not need non-linear optimization techniques, and also we don't need to impose onto the system to form canonically. Based on this fact, the subspace algorithm can equally be appropriate for both multi input-multi output (MIMO) and single input-single output (SISO) system identifications. In other words, subspace methods do not suffer from the encountered troubles and inconvenience in applying prediction error method (PEM) techniques to MIMO system identification. [3].

The differences between the subspace identification and classical identification can be seen in Fig. 1. In the classical approach, the first step is to identify the transfer function model. The next step is to obtain the state-space model using realization techniques. Also, the state vectors can be obtained using the Kalman filter. In the subspace approach, we first construct the state estimates from input-output (I/O) data by using tools of numerical linear algebra. Finally, we can obtain a state-space model by solving a least-squares problem [3].

![Fig. 1. Comparison of subspace and classical identification](image)

According to Fig. 1, an essential aspect of the subspace identification method is the estimation of the state vector from I/O data (first step in the subspace identification methods).

In recent years, MPC technology has been implemented in many applied industries such as the petroleum, chemical, and metallurgy industries. One of the major advantages of MPC controllers is the operating constraints can be represented in the optimization problem [4].

Bilinear systems are a particular category of non-linear state-dependent coefficient that have attracted the attention of many researchers. The advantage of bilinear models is that it is very similar to linear models and the accuracy of nonlinear models. Many practical system models are bilinear and more general nonlinear systems can often be well approximated by bilinear models; in other words, the bilinear systems are useful for nonlinear modeling and control [5-6].

The design of data-driven MPC approaches with guarantees on stability remains an open problem. In several recent works, learning-based or adaptive MPC schemes have been proposed, which improve an inaccurate initial model using online measurements. We offer a data-driven MPC method to control linear and bilinear systems. Our approach based on a result directly from I/O data [7-8].

The paper is organized as follows. A brief description of the subspace identification appears in Section 2. Then...
followed by the design of the MPC controller based on the subspace identification in Section 3. Section 4 presents a case study to illustrate the proposed approach. Section 5 gives the conclusion.

II. SUBSPACE IDENTIFICATION METHOD

In this section, first of all, we will describe the subspace identification method for a time-invariant MIMO linear model. Then, the subspace identification method extends to MIMO bilinear time-invariant systems.

A. Linear Subspace Identification

Consider a linear time-invariant system can be described in a state-space innovations form as [9]:

\[ x_{k+1} = Ax_k + Bu_k + Ke_k \]

\[ y_k = Cx_k + Du_k + e_k \]

(1)

where the state \( x_k \in \mathbb{R}^n \), the output \( y_k \in \mathbb{R}^l \) and the input \( u_k \in \mathbb{R}^m, e_k \in \mathbb{R}^l \) is white noise (innovations) sequence with covariance \( \Sigma_e \), and \( K \) is the steady-state Kalman filter gain.

The subspace linear predictor equation can be written as follows [9]:

\[ Y_f = H^w W_p + H^d U_f + H^s E_f \]

\[ W_p \triangleq \begin{bmatrix} Y_p \end{bmatrix} \]

(2)

Where the Indices \( f \) and \( p \) stand for the “future” and “past”. \( H^d \in \mathbb{R}^{ln \times mn} \) and \( H^s \in \mathbb{R}^{ln \times ln} \) the block Toeplitz matrices containing the impulse response of the system to deterministic input \( u_k \) and stochastic input \( e_k \) respectively which defined as:

\[ H^d = \begin{bmatrix} D & 0 & \cdots & 0 \\ CB & D & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ CA^{N-2}B & CA^{N-3}B & \cdots & D \end{bmatrix} \]

\[ H^s = \begin{bmatrix} I & 0 & \cdots & 0 \\ CK & I & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ CA^{N-2}K & CA^{N-3}K & \cdots & I \end{bmatrix} \]

(4)

The input, output and innovation block Hankel matrices are defined as:

\[ U_p = \begin{bmatrix} u_0 & u_1 & \cdots & u_{j-1} \\ u_1 & u_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ u_{N-1} & u_N & \cdots & u_{N+j-2} \end{bmatrix} \]

\[ U_f = \begin{bmatrix} u_N & u_{N+1} & \cdots & u_{N+j-1} \\ u_{N+1} & u_{N+2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ u_{2N-1} & u_{2N} & \cdots & u_{2N+j-2} \end{bmatrix} \]

(6)

Where \( U_f, U_p \in \mathbb{R}^{mn \times j} \), \( Y_f, Y_p \in \mathbb{R}^{ln \times j} \) And \( E_f, E_p \in \mathbb{R}^{kn \times j} \). \( N \) is the order of model, and \( j \) is the number of columns in data Hankel matrices. We assume that the number of columns in these matrices is \( j \to \infty \). Also, Hankel matrices defined as:

\[ U_q \triangleq \begin{bmatrix} u_0 & u_{q+1} & \cdots & u_{q+j-1} \end{bmatrix} \]

(8)

The-first term in (2) defined as:

\[ H^w W_p = \Gamma_N X_f \]

(9)

Where \( H^w \in \mathbb{R}^{ln \times (ln + mn)} \) and \( \Gamma_N \) is the extended observability matrix, and \( X_f \) is future state sequence as follows:

\[ \Gamma_N \triangleq \begin{bmatrix} C & (CA)^T & \cdots & (CA^{N-1})^T \end{bmatrix} \]

\[ X_f \triangleq \begin{bmatrix} x_N & x_{N+1} & \cdots & x_{2N-1} \end{bmatrix} \]

(10)

\[ H^d, H^s \in \mathbb{R}^{ln \times ln} \] are subspace predictor coefficients. These coefficients can be calculated using QR decomposition as follows [10]:

\[ \begin{bmatrix} U_f \\ W_p \end{bmatrix} = \begin{bmatrix} R_{11} & 0 & 0 \\ R_{21} & R_{22} & 0 \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} Q_1^T \\ Q_2^T \\ Q_3^T \end{bmatrix} = RQ^T \]

(11)

\[ L \triangleq \begin{bmatrix} R_{11} & 0 \\ R_{21} & R_{22} \end{bmatrix} \]

(12)

\[ H^d = L(\cdot, 1; i_u) \]

\[ H^s = L(\cdot, i_u + 1; i_u + i_w) \]

(13)

B. Bilinear Subspace Identification

Consider a bilinear time-invariant system can be described in a state-space innovations form as [9]:

\[ x_{k+1} = Ax_k + Nu_k \otimes S_k + Bu_k + Ke_k \]

(15)

\[ y_k = Cx_k + Du_k + e_k \]

where the matrix \( N = [N_1 \ N_2 \ \cdots \ N_m] \in \mathbb{R}^{ln \times mn} \) characterizes the bilinearity of the model. The Kronecker product \( \otimes \) of two vectors \( c \in \mathbb{R}^p \) and \( d \in \mathbb{R}^q \) is defined as \( c \otimes d = [c_1 d^T \ c_2 d^T \ \cdots \ c_p d^T]^T \in \mathbb{R}^{pq} \). The Khatri-Rao product \( \bigotimes \) of two matrices \( C = [c_1 \ c_2 \ \cdots \ c_j] \in \mathbb{R}^{kn \times j} \) and \( D = [d_1 \ d_2 \ \cdots \ d_j] \in \mathbb{R}^{kn \times j} \) is defined as the column-wise Kronecker product \( C \bigotimes D = [c_1 \otimes d_1 \ c_2 \otimes d_2 \ \cdots \ c_j \otimes d_j] \), which will be used later. The subspace bilinear predictor equation can be written as follows [9]:

\[ Y_f^b = L^b_w W_p + L^b_{w_0} U_f \otimes W_p + L^b_{d_0} U_f + L^b_{e_0} E_f \]

(16)

Where \( L^b_w, L^b_{w_0}, L^b_{d_0}, \) and \( L^b_{e_0} \) are subspace predictor coefficients. The generalized Hankel matrices can now be defined as:

\[ Y_{0j-1} \triangleq \begin{bmatrix} Y_{1j-1} \\ Y_{0j-2} \\ U_{1j-1} \otimes Y_{0j-2} \\ U_{0j-1} \otimes Y_{0j-2} \end{bmatrix}, \]

\[ U_{0j-1} \triangleq \begin{bmatrix} U_{1j-1} \\ U_{0j-2} \\ U_{1j-1} \otimes U_{0j-2} \\ U_{0j-1} \otimes U_{0j-2} \end{bmatrix}, \]

\[ E_{0j-1} \triangleq \begin{bmatrix} E_{1j-1} \\ E_{0j-2} \\ U_{1j-1} \otimes E_{0j-2} \end{bmatrix}, \]

\[ Y_p \triangleq Y_{0j-1}, Y_f \triangleq Y_{1j-1} \]

\[ U_p \triangleq U_{0j-1}, U_f \triangleq U_{1j-1} \]

(17)

(18)

(19)
\[ E_p \cong E_{0|0-1}, E_f \cong E_{0|2i-1} \]
\[ W_p \equiv W_{0|i-1} = [\varrho_{0|i-1}] \]

These coefficients can be estimated by solving the least-squares problem as follows:

\[
\begin{align*}
\min_{\theta^b, L^b_w, U^b_u} & \quad \| y_f - [I_{b^b_w} \quad L_{b^b_w} \quad U^b_u] [W_p \circ \mathcal{O} W_p] U_f \|_F^2 \\
\text{subject to} & \\
& \theta^b \in \mathbb{R}^n \\
& L^b_w \in \mathbb{R}^{n_l \times n_w} \\
& U^b_u \in \mathbb{R}^{n_l \times n_u} \\
& y_f \in \mathbb{R}^{n_f} \\
& y_{0|i} \in \mathbb{R}^{n_0} \\
& u_{0|i} \in \mathbb{R}^{n_u} \\
& \varrho_{0|i-1} \in \mathbb{R}^{n_l} \\
\end{align*}
\]  

(20)

The problem in (20) can be solved from LQ decomposition as follows:

\[
\begin{bmatrix}
U_f \\
W_p \\
U_f \circ \mathcal{O} W_p \\
Y_f
\end{bmatrix} = 
\begin{bmatrix}
R_{11} & 0 & 0 & 0 \\
R_{21} & R_{22} & 0 & 0 \\
R_{31} & R_{32} & R_{33} & 0 \\
R_{41} & R_{42} & R_{43} & R_{44}
\end{bmatrix}
\begin{bmatrix}
Q_1^c \\
Q_2^c \\
Q_3^c \\
Q_4^c
\end{bmatrix} = QR^T
\]

(21)

\[
L \triangleq [R_{41} \quad R_{42} \quad R_{43}]
\begin{bmatrix}
R_{11} & 0 & 0 \\
R_{21} & R_{22} & 0 \\
R_{31} & R_{32} & R_{33}
\end{bmatrix}^T
\]

(22)

Finally, implement the first input \( u_1 \) of the MPC control sequence \( u_{lin} \). Apply the following control step, construct the new control input vector \( \omega_p \) with the newly measured system output \( y_1 \), and repeat this step.

B. Bilinear MPC controller based on subspace identification

In the bilinear model, the predicted output sequence can be written as:

\[
\hat{y}_f = L_{b^l_w} \omega_p + L_{b^l_u} u_f + L_{u_w u_f} \circ \omega_p
\]

(27)

In other hands, NMPC cost function as follows:

\[
J = (\hat{y}_f - y_f)^T Q (\hat{y}_f - y_f) + R (u_f)
\]

(28)

By substituting (27) in (28), have a nonlinear and non-convex optimization problem. We have proposed two methods for solving this problem.

1) Gauss-Newton optimization solution

\[
\begin{bmatrix}
y_p \\
u_p
\end{bmatrix} = \begin{bmatrix}
L_{y_p}^l & L_{y_p}^b \\
L_{u_p}^l & L_{u_p}^b
\end{bmatrix} \begin{bmatrix}
y_p \\
u_p
\end{bmatrix} + \begin{bmatrix}
L_{y_p}^l & L_{y_p}^b \\
L_{u_p}^l & L_{u_p}^b
\end{bmatrix} \begin{bmatrix}
y_p \\
u_p
\end{bmatrix}
\]

(29)

From the definitions of \( u_f \) it can be directly seen that it is possible to split up \( u_f \) as:

\[
u_f = \begin{bmatrix}
u_f^b \\
u_f^l
\end{bmatrix}
\]

Where \( u_f^b \) is a vector consisting of specific products between the elements of \( u_f \). Therefore, there exists a function \( F \) that relates \( u_f \) and \( u_f^l \) in the following way:

\[
u_f = F(u_f^l)
\]

(30)

By substituting (30) in (28) have:

\[
J = (L_{u} \omega_p + L_{u} F(u_f^l) \circ \omega_p) Q (L_{u} \omega_p + L_{u} F(u_f^l) \circ \omega_p) + R (u_f^l)
\]

(31)

To solve this problem, we require the calculation of the first and second derivatives. The final Gauss-Newton optimization algorithm is:

Algorithm I

\[
\begin{align*}
\text{I.} & \quad \text{Estimate subspace predictor coefficients using data block Hankel matrices.} \\
\text{II.} & \quad \text{Set } k = 1 \text{ and } u_{f}^{1,k} = u_{f}^{1,0} \\
\text{III.} & \quad \text{Calculate the gradient } G \text{ and the Hessian } H \text{ of the cost function } J \text{ at } u_{f}^{1,k}: \\
& \quad G = \frac{\partial J(u_{f}^{1,k})}{\partial u_f^l}, \quad H = \frac{\partial^2 J(u_{f}^{1,k})}{\partial (u_f^l)^2} \\
\text{IV.} & \quad \text{Find the new solution:} \\
& \quad u_{f}^{1,k+1} = u_{f}^{1,k} - H^{-1} G
\end{align*}
\]

Iterate until the solution has converged to a local optimum \( u_{f}^{1} \). Finally, implement the first input \( u_1 \) of the MPC control sequence \( u_{lin} \). Apply the following control step, construct the new control input vector \( \omega_p \) with the newly measured system output \( y_1 \), and repeat this step.

2) Iterative approximate solution

In this method, we use an iterative Gauss-Newton optimization, which is guaranteed to converge to at least a local minimum. The generalized future inputs \( u_f^{nl} \) are defined as in:

\[
u_f^{nl} = u_{41n} = \begin{bmatrix}
u_n \\
u_{41n-1} \\
u_n \circ u_{41n-1}
\end{bmatrix}
\]

(32)

\[
\hat{y}_f^{nl} = L_{y} \omega_p + L_{y} u_f + L_{u} \circ \omega_p + L_{u} \circ u_f + L_{u} \circ u_f^{nl}
\]

(33)
\[ T_f = L_{uω}u_f \otimes ω_p + L^b_u u^b_f \]  

(34)

As shown in (33), the predictor equation consists of two parts. At every sample time, the second part is considered a constant value that it’s determined using an iterative algorithm. In other words, the nonlinear part in each step is replaced by an optimal value obtained from the previous step. Also, nonlinear optimization can be performed as quadratic optimization. By substituting (33) in (28), we have:

\[ M = (R + (L_{uω})^T Q (L_{uω}))^{-1}(L_{uω})^T \]

\[ u^i_f = \begin{bmatrix} u^i_{t+1+N-1} \\ \vdots \\ u^i_{t+1} \end{bmatrix} = -M(L_{uω}ω_p + T_f) \]  

(36)

Finally, apply the last row in (36) of the MPC control sequence \( u^f_t \), and other rows are used to construct (34) for the next sample.

**Algorithm II**

I. Estimate subspace predictor coefficients using data block Hankel matrices.

II. Set \( i = 1 \) and \( u^i_f = u^2_f \)

III. Calculate \( T_f \) using \( u^{i-1}_f \) and \( u^i_f \) using (36)

IV. If the last block-row of \( u^i_f \) satisfies the following conditions:

\[ |u^i_t - u^{i-1}_t| < δ \]

Select \( u^*_i = u^i_t \) and apply \( u^*_i \) to the plant. If the previous condition is not satisfied, set \( i = i + 1 \) and return to III [2].

**IV. SIMULATION**

Consider a deterministic bilinear systems follows [9]:

\[ x_{k+1} = \begin{bmatrix} 0.5019 & -0.3240 \\ -0.3240 & 0.0786 \end{bmatrix} x_k + \begin{bmatrix} 0.4442 \\ 0.6206 \end{bmatrix} u_k + \begin{bmatrix} -0.7164 & -0.3857 \\ -1.8002 & -0.0355 \end{bmatrix} x_k + \begin{bmatrix} 0.1 \\ 0 \end{bmatrix} e_k \]

\[ y_k = [0.9517 \\ 0.6400] x_k + 0.2473 u_k + e_k \]

The number of available input/output data set is assumed 2000. Input signal is a white zero-mean with variance 0.5 also, \( e_k \) is a white zero-mean with variance 0.01.

As it is shown in Fig. 4, Markov parameters of the bilinear model have been better estimated than the linear model because linear methods cannot describe the nonlinear system well.
As it is shown in Fig. 5. And Fig. 6. The output of MPC and NMPC at steady-state is not significantly different. While the variations of the control signal of MPC are more than that of NMPC.

As illustrated in Fig. 7. And Fig. 8. The performance of the NMPC controller is better than the MPC controller; also, the Gauss-Newton method and the iterative approximate method are not significantly different.

V. CONCLUSION

Attempts to reduce complexity in computation and combine identification and control methods have attracted the attention of many researchers. As a result, we proposed a new bilinear predictive controller design based on subspace identification to reducing the computation in system modeling. The use of subspace and bilinear modeling methods and their application in predictive control gives rise to the problem of nonlinear optimization. So, we proposed two methods for solving the nonlinear optimization problem. According to Fig. 8, it can be concluded that the methods presented are not significantly different. As a result, instead of using nonlinear optimization that is very complex, the proposed iterative algorithm method can be used.

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